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Calculation of a Certain Determinant

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Abstract: The $n\times n$ determinant $\det[(a+j-i)\Gamma(b+j+i)]$ is evaluated. This completes the calculation of the Mellin transform of the probability density of the determinant of a random quaternion self-dual matrix taken from the gaussian symplectic ensemble. The inverse Mellin transform then gives the later probability density itself.

1. Introduction and Results

Matrices whose elements, or real parameters determining the elements, are Gaussian random variables have been extensively studied for the statistical properties of their spectra [1]. One particular property is the probability density of the determinant (PDD) of such matrices. The PDD of $n \times n$ real symmetric matrices was derived by Nyquist et al. [2] to be a Meijer G-function. The PDD of $n \times n$ matrices belonging to the following classes was considered recently by Normand and one of the authors [3] (i) complex hermitian, (ii) complex, (iii) quaternion real self-dual, (iv) quaternion real, and (v) real symmetric matrices. The method used was to compute the Mellin transform of the PDD and then invert it to get the PDD itself. The Mellin transform in case (i) turned out to be a determinant with elements $\Gamma(a+i+j)$, simple enough to evaluate. Thus the PDD for matrices of cases (i) and (ii) was found to be either a Meijer G-function or a linear combination of two Meijer G-functions. For case (iii) the Mellin transform of the PDD was found to depend on determinants of matrices with elements $[(a+j-i)\Gamma(b+j+i)]$ and $a = 0$ or $1/2$, $b = s/2$ or $s \pm 1/2$. These determinants are evaluated here thus completing the derivation of the PDD of matrices in case (iii). The question of the PDD of matrices in cases (iv) and (v) remains open.

In this brief note we evaluate the determinant of any matrix

$$[(a+j-i)\Gamma(b+j+i)]_{i,j=0,1,\ldots,n-1} \tag{1}$$

or its pfaffian when $a = 0$. The result is

$$M_n(a,b) = \det[(a+j-i)\Gamma(b+j+i)]_{i,j=0,1,\ldots,n-1} \tag{2}$$

$$= D_n \prod_{i=0}^{n-1} i!\,\Gamma(b+i), \tag{3}$$

$$\sqrt{M_{2n}(0,b)} = \mathrm{Pf}[(j-i)\Gamma(b+j+i)]_{i,j=0,1,\ldots,2n-1}$$
$$= \prod_{i=0}^{n-1}(2i+1)!\,\Gamma(b+2i+1), \tag{4}$$

where

$$D_n = \det[a\delta_{i,j} - \delta_{i,j+1} + j(b+i)\delta_{i,j-1}] \tag{5}$$
$$= \text{coeff. of } (z^n/n!) \text{ in } (1-z)^{-(b+a)/2}(1+z)^{-(b-a)/2} \tag{6}$$
$$= \sum_{k=0}^{n}(-1)^k \left(\frac{b-a}{2}\right)_k \left(\frac{b+a}{2}\right)_{n-k} \tag{7}$$

with Pochhammer's symbol

$$(a)_n = \Gamma(a+n)/\Gamma(a). \tag{8}$$

The expression for D_n simplifies when a is a small integer or when $a=b$.

The final result for the probability density $g_n(y)$ of the determinant of an $n\times n$ random quaternion self-dual matrix is as follows. It is convenient to consider the even and odd parts $g_{n\pm}(y) = \frac{1}{2}[g_n(y) \pm g_n(-y)]$ of $g_n(y)$ separately. The Mellin transform of the even part $g_{n+}(y)$ was found [3] to be a constant times $M_n\left(\frac{1}{2}, \frac{s}{2}\right)$ while that of the odd part $g_{n-}(y)$ was found to be a constant times $\mathrm{Pf}[(j-i)\Gamma\left(j+i+\frac{s+1}{2}\right) \times \mathrm{Pf}[(j-i)\Gamma\left(j+i+\frac{s-1}{2}\right)$, if n is even, and zero if n is odd. The Mellin transforms of $g_{n\pm}(y)$ are thus seen to be either a product of Gamma functions or a linear combination of these products. The $g_{n\pm}(y)$ and their sum $g_n(y)$ themselves are thus either a Meijer G-function or a linear combination of two Meijer G-functions. However, it is cumbersome to write their expressions except for small values of n.

The corresponding problem for random real symmetric matrices or for quaternion real matrices with gaussian element densities remains open, as noted earlier.

2. Evaluation of the Determinant

We will need the following lemma.

Lemma. *For j a non-negative integer and A and B complex numbers one has the identity*

$$F(j,A,B) \equiv \sum_{k=0}^{j}(-1)^k\binom{j}{k}(A+j-k)_k(B)_{j-k}. = (B-A-j+1)_j \tag{9}$$

Proof. The lemma is trivial for $j = 0$ and easy to verify for $j = 1$. Suppose that it is true for some positive integer j. Then for the next integer

$$\begin{aligned} F(j+1, A, B) &= \sum_{k=0}^{j+1} (-1)^k \binom{j+1}{k} (A+j+1-k)_k (B)_{j+1-k} \\ &= \sum_{k=0}^{j+1} (-1)^k \left[\binom{j}{k} + \binom{j}{k-1} \right] (A+j+1-k)_k (B)_{j+1-k} \\ &= B \sum_k (-1)^k \binom{j}{k} (A+1+j-k)_k (B+1)_{j-k} \\ &\quad - (A+j) \sum_k (-1)^{k-1} \binom{j}{k-1} (A+j-k+1)_{k-1} (B)_{j-k+1} \\ &= B\ F(j, A+1, B+1) - (A+j)\ F(j, A, B) \\ &= (B-A-j) F(j, A, B) = (B-A-j)_{j+1}. \end{aligned} \tag{10}$$

And so it is true for every positive integer j. Note that the binomial coefficient $\binom{j}{k}$ is zero if the integer k is either negative or is greater than j. Actually, $k! = \Gamma(k+1)$ can be replaced by ∞ whenever k is a negative integer. □

Corollary.

$$\sum_{k=0}^{j} (-1)^k \binom{j}{k} (j-k)(A+j-k)_k (B)_{j-k} = jB(B-A-j+2)_{j-1}. \tag{11}$$

Proof.

$$\begin{aligned} \text{Left hand side } &= jB \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} (A+j-k)_k (B+1)_{j-1-k} \\ &= jBF(j-1, A+1, B+1) = jB(B-A-j+2)_{j-1}. \end{aligned} \tag{12}$$

□

Determinant. The determinant of a matrix is not changed if we add to any of its rows (columns) an arbitrary linear combination of the other rows (columns). Replacing the j^{th} column $C_j = (a+j-i)\Gamma(b+j+i)$ by

$$C'_j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{\Gamma(b+j)}{\Gamma(b+j-k)} C_{j-k} \tag{13}$$

$$= (a-i) \sum_{k=0}^{j} (-1)^k \binom{j}{k} (b+j-k)_k \Gamma(b+i+j-k)$$

$$+ \sum_{k=0}^{j} (-1)^k \binom{j}{k} (j-k)(b+j-k)_k \Gamma(b+i+j-k)$$

$$= \Gamma(b+i)[(a-i)F(j,b,b+i) + j(b+i)F(j-1,b+1,b+i+1)]$$
$$= \Gamma(b+i)[(a-i)(i-j+1)_j + j(b+i)(i-j+2)_{j-1}]$$
$$= \Gamma(b+i)\left[(a-i)\frac{i!}{(i-j)!} + j(b+i)\frac{i!}{(i-j+1)!}\right], \tag{14}$$

where in the third line above we have used the lemma and its corollary. Taking out the factors $\Gamma(b+i)$ one has

$$M_n(a,b) = \left[\prod_{i=0}^{n-1}\Gamma(b+i)\right]\det\left[(a-i)\frac{i!}{(i-j)!} + j(b+i)\frac{i!}{(i-j+1)!}\right]. \tag{15}$$

Now replace the row

$$R_i = \left[(a-i)\frac{i!}{(i-j)!} + j(b+i)\frac{i!}{(i-j+1)!}\right] \tag{16}$$

by the linear combination

$$R_i' = \sum_{k=0}^{i}(-1)^k\binom{i}{k}R_{i-k} \tag{17}$$
$$= \sum_{k=0}^{i}(-1)^k\binom{i}{k}\left[(a-i+k)\frac{(i-k)!}{(i-k-j)!} + j(b+i-k)\frac{(i-k)!}{(i-k-j+1)!}\right]$$
$$= i!\sum_{k=0}^{i}(-1)^k\left[\frac{a-i}{k!(i-j-k)!} + \frac{1}{(k-1)!(i-j-k)!}\right.$$
$$\left. + \frac{j(b+i)}{k!(i-k-j+1)!} - \frac{j}{(k-1)!(i-k-j+1)!}\right]$$
$$= i!\left[\frac{(a-i)}{(i-j)!}\sum_{k=0}^{i}(-1)^k\binom{i-j}{k} - \frac{1}{(i-j-1)!}\sum_{k=0}^{i}(-1)^{k-1}\binom{i-j-1}{k-1}\right.$$
$$\left. + \frac{j(b+i)}{(i-j+1)!}\sum_{k=0}^{i}(-1)^k\binom{i-j+1}{k} + \frac{j}{(i-j)!}\sum_{k=0}^{i}(-1)^{k-1}\binom{i-j}{k-1}\right]$$
$$= i!\,[(a-i)\delta_{i,j} - \delta_{i,j+1} + j(b+i)\delta_{i,j-1} + j\delta_{i,j}]$$
$$= i!\,[a\delta_{i,j} - \delta_{i,j+1} + j(b+i)\delta_{i,j-1}]. \tag{18}$$

Thus

$$M_n(a,b) = D_n\prod_{i=0}^{n-1} i!\,\Gamma(b+i), \tag{19}$$

where

$$D_n = \det[a\delta_{i,j} - \delta_{i,j+1} + j(b+i)\delta_{i,j-1}]. \tag{20}$$

Expanding by the last row and last column, one gets the recurrence relation

$$D_{n+1} = aD_n + n(b+n-1)D_{n-1}, \tag{21}$$

$$D_0 = 1, \quad D_1 = a, \quad D_2 = a^2 + b, \quad \ldots . \tag{22}$$

To find D_n introduce the generating function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} D_n. \tag{23}$$

Multiplying Eq. (21) on both sides by $z^n/n!$ and summing over n from 0 to ∞, one has

$$f'(z) = af(z) + bzf(z) + z^2 f'(z) \tag{24}$$

since

$$\sum_{n=0}^{\infty} D_{n+1} \frac{z^n}{n!} = \frac{d}{dz} \sum_{n=0}^{\infty} D_n \frac{z^n}{n!} \tag{25}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} n(b+n-1)D_{n-1} \frac{z^n}{n!} &= bz \sum_{n=1}^{\infty} D_{n-1} \frac{z^{n-1}}{(n-1)!} + z^2 \sum_{n=2}^{\infty} D_{n-1} \frac{z^{n-2}}{(n-2)!} \\ &= bzf(z) + z^2 f'(z) \end{aligned} \tag{26}$$

hence

$$(1-z^2) f'(z) = (a+bz) f(z) \tag{27}$$

or

$$f(z) = (1-z)^{-(b+a)/2} (1+z)^{-(b-a)/2}. \tag{28}$$

This gives Eqs. (3) and (7).

If $a = 0$,

$$f(z) = (1-z^2)^{-b/2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} \left(\frac{b}{2}\right)_n \tag{29}$$

so that the determinant (2) of an $n \times n$ matrix is zero when n is odd, as it should, and when n is even it is

$$\begin{aligned} M_{2n}(0,b) &= \frac{(2n)!}{n!} \left(\frac{b}{2}\right)_n \prod_{i=0}^{2n-1} i!\, \Gamma(b+i) \\ &= \left[\prod_{i=0}^{n-1} (2i+1)!\, \Gamma(b+2i+1) \right]^2 . \end{aligned} \tag{30}$$

The pfaffian is the square root of this and its sign can be fixed by looking at one of the terms.

When $a = b$,

$$f(z) = (1 - z)^{-a} \quad \text{and} \quad D_n = (a)_n. \tag{31}$$

When a is a small integer, expression (7) for D_n simplifies. For example, when $a = 1$,

$$f(z) = (1 + z)(1 - z^2)^{-(b+1)/2} = (1 + z) \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} \left(\frac{b+1}{2}\right)_n \tag{32}$$

and

$$D_{2n} = \frac{(2n)!}{n!} \left(\frac{b+1}{2}\right)_n, \tag{33}$$

$$D_{2n+1} = \frac{(2n+1)!}{n!} \left(\frac{b+1}{2}\right)_n. \tag{34}$$

Or when $a = 2$,

$$f(z) = (1 + z)^2(1 - z^2)^{-(b+2)/2} = (1 + z)^2 \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} \left(\frac{b+2}{2}\right)_n \tag{35}$$

and

$$D_{2n} = \frac{(2n)!}{n!} \left(\frac{b+2}{2}\right)_n + \frac{(2n)!}{(n-1)!} \left(\frac{b+2}{2}\right)_{n-1}, \tag{36}$$

$$D_{2n+1} = 2. \frac{(2n+1)!}{n!} \left(\frac{b+2}{2}\right)_n. \tag{37}$$

etc.

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