An Improved Iterative SENSE Reconstruction Method

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ABSTRACT: The conjugate-gradient (CG)-based non-Cartesian SENSE reconstruction usually exhibits unstable convergence behavior due to the ill conditioning of the generalized encoding matrix (GEM). To overcome this difficulty, an improved iterative SENSE approach is presented. During a so-called Lanczos iteration process, which is equally efficient as CG, the inversion of GEM can be gradually approximated by calculating inversions of a series of small tridiagonal matrices. In this fashion, inner regularization can be incorporated into the reconstruction without touching the iteration process. The degree of regularization can be determined based on the eigenvalue information provided by the Lanczos process. With inner regularization adaptively applied for every iteration vector, the convergence behavior of iterative SENSE can be significantly improved and noise amplification can be avoided. The feasibility of this novel iterative SENSE technique is demonstrated by radial and spiral MRI experiments. © 2007 Wiley Periodicals, Inc. Concepts Magn Reson Part B (Magn Reson Engineering) 31B: 44–50, 2007

KEY WORDS: parallel MRI; SENSE; non-Cartesian trajectories; iterative reconstruction; regularization

INTRODUCTION

In parallel imaging, the difference in coil sensitivities between individual coil elements in a receive array is exploited for spatial encoding as a supplement to conventional gradient encoding. In this way, the number of gradient encoding steps can be reduced and, consequently, the MRI scanning can be accelerated. In contrast to conventional pure Fourier reconstruction, parallel MRI reconstruction inevitably involves inversion problems due to the mixed encoding scheme. This brings two significant concerns in image reconstruction. First, computational complexity is increased dramatically because inversion can no longer be performed merely by FFT; second, the problem of ill conditioning arises because coil sensitivities encoding in general introduces irregularity into the encoding matrix. For regularly undersampled Cartesian k-space trajectories, a number of practical and efficient strategies have been reported to minimize these difficulties during the past few years, such as sensitivity encoding (SENSE) (*1*), simultaneous acquisition of spatial harmonics (SMASH) (*2–5*), parallel imaging with localized sensitivities (PILS) (*6*), sensitivity profiles from an array of coils for encoding and reconstruction in parallel (SPACE RIP) (*7*), and generalized autocalibrating partially parallel acquisition (GRAPPA) (*8*). In various manners and to different extents, these techniques can successfully "decouple"

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the sensitivity encoding from Fourier encoding, and then decoding can be performed by combination of FFT and inversion of small-size matrices. However, when parallel MRI is employed with non-Cartesian trajectories, reconstruction is much more a problem because it is difficult to separate different encoding schemes, and straightforward inversion of the generalized encoding matrix (GEM) is seriously hindered by its numerical complexity.

The iterative SENSE reconstruction strategy proposed by Pruessmann and Kannengiesser (*9–11*) is an advanced parallel imaging technique for arbitrary k-space trajectories. The basic idea of iterative SENSE is to invert the large GEM using the conjugate gradient (CG) iteration method and to implement the most time-demanding matrix vector multiplication with a highly efficient gridding/FFT process. Using this CG-based iterative technique, numerical complexity of the inversion of GEM can be greatly reduced and reconstructions become feasible. However, the ill-conditioning problem has not been well addressed in Pruessmann's report. In general, ill conditioning of the GEM affects the convergence behavior of CG iteration and causes instability in reconstruction. More specifically, in many cases the iteration exhibits a semiconvergence behavior, that is, the iteration vector initially approaches the desired exact solution and then, in later stages of the iteration, converges to some other undesired vector.

It is well known that ill-conditioning problems can be overcome or mitigated by regularization tools. In previous works, some popular regularization techniques, such as Tikhonov regularization and truncated singular value decomposition (SVD), have been applied in SENSE and SMASH to improve SNR of Cartesian parallel imaging (*3*, *12*, *13*). Recently, methods of performing explicit Tikhonov regularization with generalized CG-SENSE were also reported (*14*).

To address the ill-conditioning problem in non-Cartesian SENSE, in this study an improved iterative reconstruction technique is proposed. During a so-called Lanczos iteration process, which is equally efficient as CG, the inversion of GEM can be gradually approximated by calculating inversions of a series of small tridiagonal matrices (*15–17*). In this fashion, regularization can be easily incorporated into the reconstruction. Because regularization can be flexibly employed in each iteration loop independently, this strategy is known as inner regularization. Inner regularization does not affect the iteration process and the convergence behavior. Therefore, it can stabilize the reconstruction without increasing the running time. Mathematic principles and implementation issues of this improved iterative SENSE technique are described in this article. The effect of inner regularization to stabilize the reconstruction is demonstrated by phantom and in vivo MRI experiments.

THEORY

Review of CG-Based Iterative SENSE Reconstruction

Generally, parallel imaging reconstruction can be simply formulated as a linear equation system

$$s = Em, \qquad [1]$$

where s is the vector of signal samples, m is the vector of the unknown image, and E is the GEM composed of gradient encoding and coil sensitivity encoding.

The least squares estimate of Eq. [1] can be obtained by solving the normal equation, which reads:

$$(E^H E)m = E^H s, \qquad [2]$$

where the superscript H denotes the complex conjugate transpose.

Directly solving Eq. [2] is numerically prohibitive, instead, it is feasible to perform reconstruction iteratively (*9–11*). Using the highly efficient CG iteration scheme, the coefficient matrix is only accessed via matrix-vector multiplications, and a sequence of iteration vectors is produced that converge to the least squares solution. In each CG loop, the most time-consuming matrix-vector multiplication can be efficiently implemented by a combination of gridding principle and FFT. Furthermore, for faster convergence, optional density and intensity correction can be introduced to precondition the coefficient matrix.

The main drawback of this method is its convergence behavior. Usually, the coefficient matrix is rather large and notoriously ill conditioned due to the mixed encoding scheme. As a consequence, the iteration process does not converge stably to the exact solution. Typically, in early stages of iteration the reconstruction goes well, while noise amplification tends to arise and become more and more serious after a number of iteration loops. This phenomenon hinders automatic implementation of the non-Cartesian SENSE technique.

Lanczos Iteration Process

In this work we propose a novel iterative SENSE reconstruction strategy based on Lanczos method.

Simply speaking, the principle of this algorithm is to project a large Hermitian matrix onto a set of suitably chosen orthogonal vectors by an iteration process so that it is reduced to a much smaller $(j + 1) \times (j + 1)$ matrix with j denoting the iteration count. In SENSE reconstruction, Lanczos method is applied to the normal equation form in Eq. [2]. Let $A = E^H E$ is an n-by-n Hermitian matrix; the stepwise nature of the Lanczos process results in a tridiagonal matrix T_j and in a unitary matrix $Q_j = [q_1, q_2, \ldots q_j]$ after j steps of the process, which are related as

$$AQ_j = Q_j T_j + \beta_j q_{j+1} e_j^T \qquad [3]$$

where e_j is the jth axis vector. β_j decreases with j and $\beta_j \rightarrow 0$ when $j \rightarrow rank(A)$. An explicit algorithm of Lanczos method is stated in the appendix.

The Lanczos iteration process converges as fast as the CG method (i.e., β_j decreases to zero quickly). In practice, after a small number of iteration loops ($j \ll n$), β_j becomes numerically negligible, such that Eq. [3] becomes

$$A = Q_j T_j Q_j^H$$

and since Q_j is a unitary matrix, inverse of A can be simplified as

$$A^{-1} = Q_j (T_j)^{-1} Q_j^H \qquad [4]$$

let

$$m_j = Q_j (T_j)^{-1} Q_j^H E^H s \qquad [5]$$

from Eq. [3] and Eq. [4] we see that m_j approaches the least squares solution m_{LS} along with the iteration.

Because $j \ll n$, inverse of T_j is numerically cheap. Therefore, almost all of the computation effort lies in the Lanczos iterations, which, like the CG method, involve matrix-vector multiplications. Following (*11*), in our implementation this is also accomplished by combination of FFT and gridding principle.

Numerically, the Lanczos process is equally efficient as a typical CG algorithm (*15*). However, the Lanczos method holds a unique desirable property: the eigenvalues, or singular values (SV), of A are gradually captured in decreasing order by the small tridiagonal matrix T_j, that is, the j eigenvalues of T_j are approximations of the j largest eigenvalues of A, with higher accuracy for larger eigenvalues and more iteration cycles (*15*). This provides the possibility to control the iteration progress by monitoring the eigenvalues of the resulted T_j. Along the progress of iteration, large eigenvalues are more accurately captured, while more small eigenvalues appear. Stopping criterion can then be established, for example, based on max(SV)/min(SV), which reflects the condition of T_j, as iteration should be stopped before T_j becomes too much ill posed. In addition, the eigenvalue information can be used as a reference to decide the degree of regularization applied to the inversion, because regularization is essentially manipulating the SVs of the matrix to be inverted.

Inner Regularization

Taking advantage of the property of the Lanczos process, regularization can be applied only for inversion of T_j in Eq. [5]. Because the eigenvalues of A are captured by T_j in decreasing order, regularization by directly manipulating the SV components in $(T_j)^{-1}$ is obvious. In this study we set a SV threshold and simply disregard all the SV components below that threshold.

Assume $T_j = \sum_{i=1}^{j} u_i \sigma_i v_i^T$ is its SVD, then $(T_j)^{-1} = \sum_{i=1}^{j} v_i \sigma_i^{-1} u_i^T$, regularization by SVD truncating yields $(T_j)_{(reg)}^{-1} = \sum_{\sigma_i > threshold} v_i \sigma_i^{-1} u_i^T$. Throughout this study, the SV threshold is set to 1% of the largest SV.

With inner regularization, the intermediate solution for the j-th iteration then reads

$$m_j = Q_j (T_j)_{(reg)}^{-1} Q_j^H E^H s.$$

Implementation

This novel method is based on Lanczos iteration process and inner regularization. Lanczos process provides the possibility to apply inner regularization into the inverse without touching the iterations; meanwhile, it provides eigenvalue information to determine the degree of regularization. With inner regularization adaptively applied for every iteration vector, the process converges stably to the regularized solution. In practice, to avoid waste of computation cost, the iteration is stopped when max(SV)/min(SV) exceeds a condition number threshold (set to 300 in this study). Then regularization was applied to the final iteration result by eliminating the small eigenvalue components.

In summary, this improved iterative SENSE technique consists of three key procedures: Lanczos process, SVD of T_j, and regularization to the final iteration result. The flowchart of implementing this method is shown in Fig. 1, where the gridding/FFT

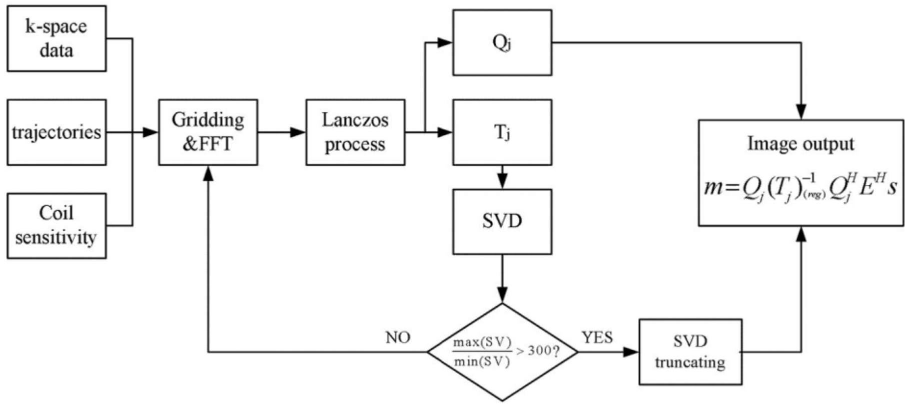

Figure 1 Implementation flowchart of the improved iterative SENSE reconstruction.

process, which is equivalent to multiplication by $E^H E$, is adopted from Fig. 1 in (*11*).

METHODS

MRI Experiments

To demonstrate the performance of this improved non-Cartesian SENSE reconstruction method, two most popular non-Cartesian trajectories, spiral and radial trajectories, were used in this work. Radial experiments were performed on a 1.5T SIEMENS Symphony system, and spiral data were acquired on a Siemens 3T Trio scanner. In either case, a body coil was used as transmit coil and an eight-element head coil array was employed as receive coil.

Transverse phantom imaging was conducted with a homogeneous sphere phantom. For radial acquisition, a siemens true-FISP radial sequence was used. A total of 128 projections were acquired with 256 samples in each projection. The acquisition parameters were FOV = 200 × 200 mm, slice thickness = 5 mm, TE = 3.4 ms, TR = 53.5 ms, flip angle = 90°, matrix = 128 × 128. For spiral acquisition, a 2D spiral sequence was used (FOV = 200 × 200 mm^2, slice thickness = 3 mm, TR = 3 s, flip angle = 80°). A full dataset with four interleaves for 128 × 128 imaging was acquired. In the readout direction, 3,908 samples were acquired for each interleaf with standard sampling density.

Axial brain MRI data were acquired using the same radial and spiral sequences. For radial acquisition, 256 projections were scanned with 256 samples per projection. FOV = 220 × 220 mm, slice thickness = 5 mm, TE = 3.4 ms, TR = 53.5 ms, flip angle = 70°, matrix = 128 × 128. For spiral acquisition, the scanning parameters were the same as in the phantom imaging.

Reconstruction

Iterative SENSE reconstructions based on the Lanczos process were performed with the radial and spiral data. For phantom data, conventional CG-based reconstructions were also performed for comparison. Matrix-vector multiplications were performed using the gridding/FFT procedure, whereas gridding was implemented using the LS-nuFFT method (*18*, *19*). Density correction was applied in the iteration loops for preconditioning. The full datasets were evenly decimated in the interleaf or projection direction to simulate accelerated cases. The sensitivity profiles were calibrated using the densely sampled central regions of the radial and spiral k-spaces, as described in (*20*). All algorithms were implemented offline on a Pentium M 1.5 GHz PC with 512 MB RAM.

RESULTS

The reconstructed phantom image quality from 8X accelerated radial data (16 projections) and 2X accelerated spiral data (two interleaves) varying with iteration count is shown in Figs. 2(a) and 2(b), respectively. The image quality (reconstruction error) is measured by

$$err(j) = \sum_n \left|(I_n(j) - I_n^{ref})\right|,$$

where $I_n(j)$ is the image resulted after j iterations with n denoting the pixel index; I_n^{ref} is the reference image reconstructed from the corresponding full datasets.

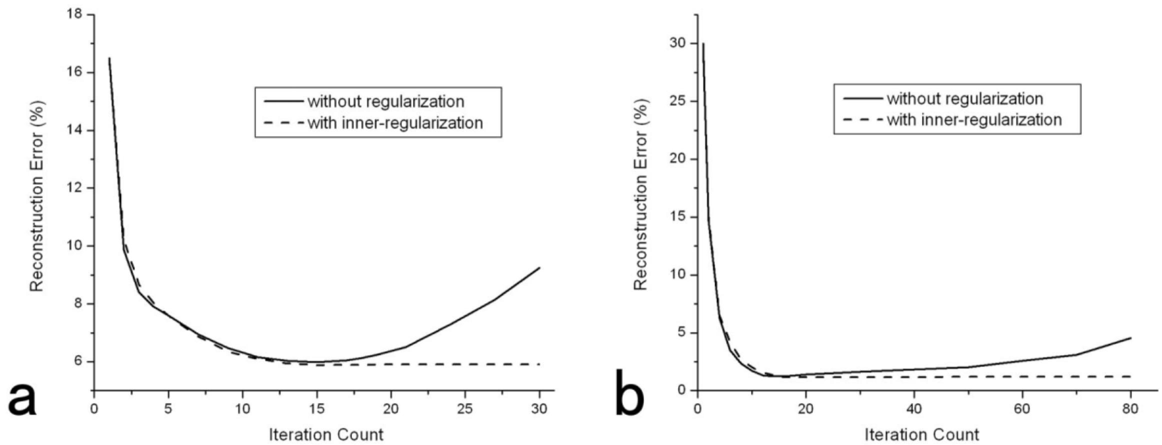

Figure 2 Plot of reconstruction errors against the iteration count for SENSE reconstructed phantom images with (a) radial trajectories and (b) spiral trajectories. The solid lines show the results of conventional CG-based reconstruction; the dashed lines represent the results using Lanczos process with inner regularization applied in every iteration.

The solid lines are the result of conventional CG-based method, and the dashed ones are their counterparts using the Lanczos process with inner regularization. The curves clearly show that without regularization, the CG iteration does not converge stably. The image quality improves with the iterations in early stages but deteriorates in later stages. With the inner-regularization strategy applied in every loop, the convergence behavior of the reconstruction is significantly improved, as shown by the dashed lines.

Figure 3 shows the 8X accelerated radial phantom images after 30 iterations using the conventional CG method (see Fig. 3a) and using inner regularization (see Fig. 3b), respectively. Observe that the resulted image after 30 CG iterations is seriously contaminated by noise, whereas the inner regularization results exhibit excellent compromise between noise and artifacts. Similar facts can be observed in Fig. 4, which correspond to the 60-iteration reconstruction of 2X accelerated spiral phantom imaging.

The improved iterative SENSE reconstructions with automatic stopping rule and inner regularization were implemented with in vivo data, and the results are shown in Figs. 5–7. Figure 5 shows the evolving of some singular values of T_j in the Lanczos processes of the reconstructions of radial (see Fig. 5a) and spiral (see Fig. 5b) head images. The solid lines show the largest five SVs and the dotted lines represent the smallest SVs. As predicted in the theory section, SVs are gradually grasped in decreasing order of magnitude. The large SVs are more accurately captured; meanwhile, more small SVs appear with the iteration, showing worsening condition in the progress of the reconstruction. For the 8X SENSE radial reconstruction, max(SV)/min(SV) exceeded 300 after 25 iterations, where the Lanczos process was stopped, and the resulted image is shown in Fig. 6(a), which suffers from minor noise amplification. Inner regularization

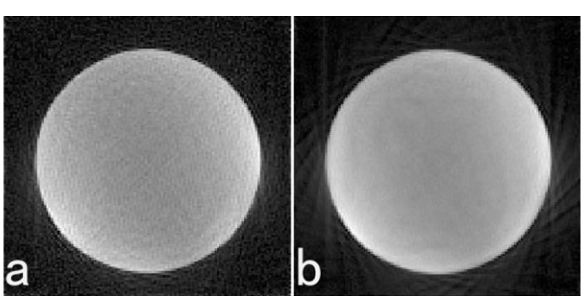

Figure 3 Thirty-iteration reconstructions of 8X accelerated radial phantom images using the conventional CG-based method (a) and using the inner-regularization technique (b).

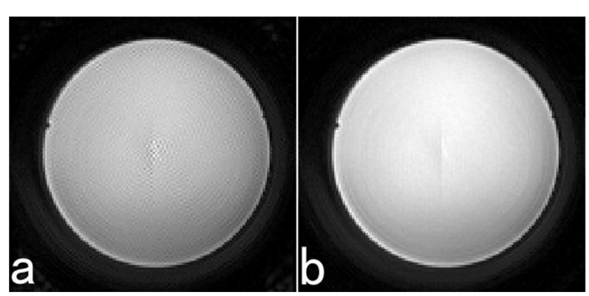

Figure 4 Sixty-iteration reconstructions of 2X accelerated spiral phantom images using the conventional CG-based method (a) and using the inner-regularization technique (b).

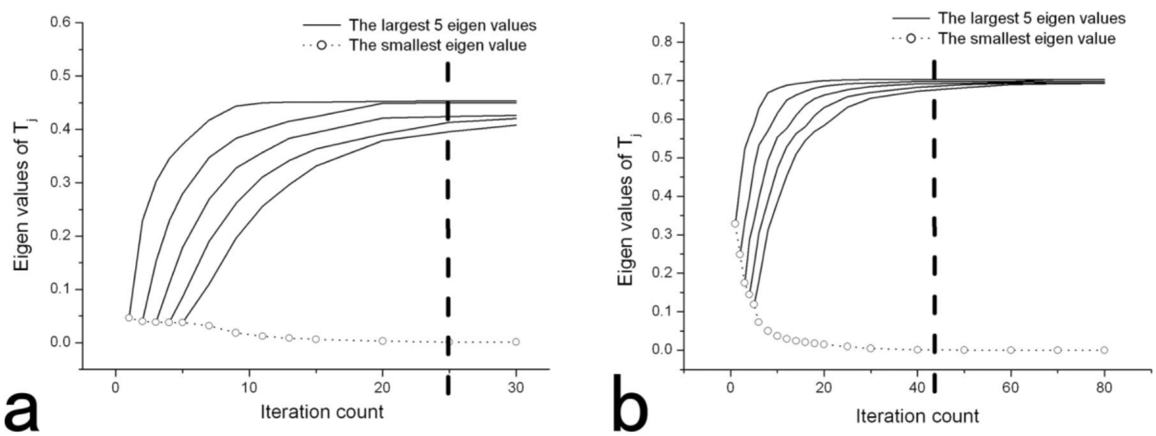

Figure 5 Evolution of some singular values of T_j in the Lanczos processes for the reconstructions of radial (a) and spiral (b) head images. The solid lines show the largest five SVs, and the dotted lines represent the smallest SVs. The vertical dashed lines mark the positions where the process should be stopped based on the max(SV)/min(SV) criterion (25 iterations for the radial case and 43 iterations for the spiral case).

is then applied, and the final image is shown in Fig. 6(b). For the 2X SENSE spiral reconstruction, the process was stopped after 43 iterations. Corresponding results with and without inner regularization are shown in Figs. 7(a) and 7(b), respectively.

Comparing the two pairs in Figs. 6 and 7, it can be observed that SNR is significantly improved by inner regularization at a modest cost of artifacts increase. This demonstrates that the inner regularization technique proposed in this study can automatically achieve good compromise between SNR and artifacts, thus improving the overall image quality for non-Cartesian SENSE imaging.

DISCUSSION

It has been demonstrated that inner regularization can stabilize the iteration SENSE reconstruction and improve the SNR of the final images. In our experience, quality of non-Cartesian SENSE reconstruction is highly dependent on the degree of regularization. Overregularization tends to cause high level artifacts, whereas underregularization cannot prevent the noise booming in later iterations. The importance of Lanczos process in our strategy is that it provides eigenvalue information, which helps to apply inner regularization in an appropriate extent adaptively.

CONCLUSION

An improved algorithm for iterative SENSE reconstruction has been proposed. Based on the Lanczos iteration process, inner regularization can be applied adaptively to stabilize the reconstruction and avoid noise amplification. In practice, this algorithm is im-

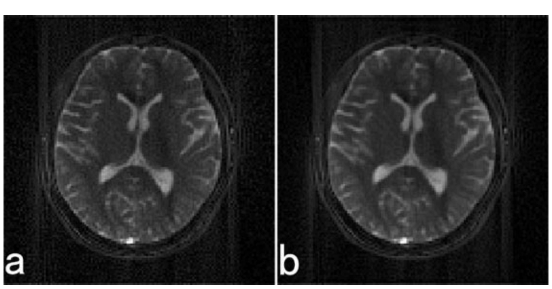

Figure 6 Iterative reconstructions of 8X-SENSE radial head images. (a) Immediate result upon the stop of iteration without regularization. (b) Result with truncated SVD regularization applied after iteration stops.

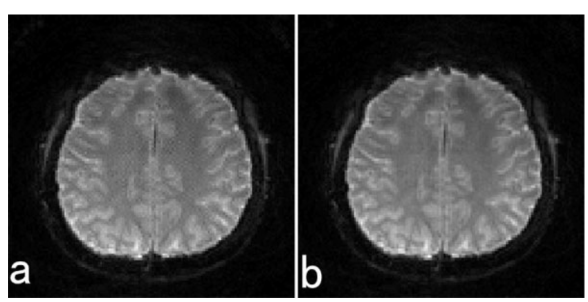

Figure 7 Iterative reconstructions of 2X-SENSE spiral head images. (a) Immediate result upon the stop of iteration without regularization. (b) Result with truncated SVD regularization applied after iteration stops.

plemented as a combination of Lanczos process, SVD monitoring, and SVD truncating. The feasibility of this novel iterative SENSE technique has been demonstrated by radial and spiral MRI experiments.

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APPENDIX

Given a starting vector b and a subroutine for matrix-vector multiplication $y = Ax$ for any x, where A is an n-by-n Hermitian matrix. The following algorithm computes a tridiagonal matrix

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & 0 \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ 0 & & \beta_{n-1} & \alpha_n \end{bmatrix}$$

and a unitary Q such that $A = QTQ^H$ (*16*).

$q_0 = 0; \beta_0 = 0;$
$q_1 = b/\|b\|_2;$
for $j = 1$ *to* n
$y = Aq_j;$
$\alpha_j = q_j^H y;$
$y = y - \alpha_j q_j - \beta_{j-1} q_{j-1};$
$\beta_j = \|y\|_2;$
if $\beta_j = 0$, *quit*; *end*
$q_{j+1} = y/\beta_j;$
end

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